Springer

Journal of Mechanical Science and Technology

Journal of Mechanical Science and Technology 23 (2009) 2612~2623

www.springerlink.com/content/1738-494x
DOI 10.1007/s12206-009-0505-2

Uncertainty estimation of reliability redundancy in complex systems based on the Cross-Entropy method †

Gui-Bao Wang[1,2], Hong-Zhong Huang[1,*], Yu Liu[1], Xiaoling Zhang[1] and Zhonglai Wang[1]

[1]*School of Mechatronics Engineering, University of Electronic Science and Technology of China, Chengdu, Sichuan, 610054, China*
[2]*PLA representative office stationed at factory 372, Jingdezhen, Jiangxi, 333002, China*

(Manuscript Received February 12, 2008; Revised September 18, 2008; Accepted May 3, 2009)

Abstract

The article aims to estimate the uncertainty of possible failure events of redundancy systems based on the cross-entropy (CE) method. Failure events of subsystems and components always result in the incomplete or complete failure of engineering systems, yet optimal condition monitoring of a complex system is heavily dependent on the accuracy analysis of all the failure events of subsystems and components and their interaction effects. The CE method is a versatile tool for estimating probabilities of rare events in complex systems with the least bias beyond conditional constraints. In this paper we introduce the CE method for analyzing the system reliability with the highest uncertainty among all possibilities satisfying supplied moment constraints, and developed numerical CE algorithms capable of estimating the uncertainty of failure modes in an M-dimensional redundancy system domain with moment constraints of order up to N. A general computational framework of event estimation and condition monitoring of redundancy systems is illustrated in which the Monte Carlo simulations and CE optimization algorithms are combined. Numerical results indicate potential improvements in the measure of the uncertainty of redundancy systems that would lead to the best-fit analysis of all the complete or incomplete failure events.

Keywords: Reliability; Redundancy; Complex systems; Cross-Entropy

1. Introduction

There are many uncertainties concerning the failure modes of complex systems which may consist of many components and/or different subsystems. Causes of uncertainties may be interrelated and introduce dependencies, while ignoring these dependencies may lead to large errors. The states of a system are represented by a system and by subsystems of random events in different relations and on various levels [1-2]. In practice, reliability prediction of an engineering complex system is often viewed as object analysis of many discrete interacting conditional failures of subsystems in different ways [3-7]. The goals of such analysis are to determine the whole cumulated effects and failure modes on the overall behavior of the system. Accuracy condition indicators for robust fault detection [8-10] and subset simulation for reliability sensitivity analysis [3, 5, 11, 12] make the majority of the system states be observable. Nevertheless, the uncertainties of system failure modes may be considered at another level. In event-oriented system analysis a system is defined not only by its physical components, but also by its all or at least known or important states [2]. There are some states that can be in common with several system features that may be unobservable, undefined or unknown. One of the most basic, useful approaches to eliminate the uncertainties of system failure possibilities is adopting redundancy optimization. The redundancy optimization

† This paper was recommended for publication in revised form by Associate Editor Jeong Sam Han
*Corresponding author. Tel.: +86 28 8320 6916, Fax.: +86 28 8320 6828
E-mail address: hzhuang@uestc.edu.cn

problem has been addressed in a number of studies [13-16], and components or subsystems redundancy can be viewed in the majority of complex engineering systems in different ways. The redundancy problem of a system with different elements may be considered as a problem of multi-state series–parallel system structure optimization [15], and the method has been extended to systems with bridge topology (see for example Ref. [16]).

Probability estimation of a system provides a qualitative or quantitative description of the likely occurrence of a particular event. The probability of an event has been defined as its long-run relative frequency. A subjective probability describes an individual's personal judgment about how likely a particular event is to occur [17]. Probability is conventionally expressed on a scale from 0 to 1, and a rare event has a probability close to 0. A person's subjective probability of an event describes his/her degree of belief in the event. It is not based on any precise computation but is often thought as a personal degree of belief that a rare event in a complex system will occur [18]. For complex systems, researches and decision makers pay more attention to the performance residual capacities, and it is unrealistic to take into account all the effects of varying subsystems non-operational conditions. In some experiments of complex systems, all outcomes are equally likely. The value of the system operational objective function and the uncertainties of system failure possibilities based on the probabilistic theory become difficult or impractical to evaluate.

Another useful approach to estimate the uncertainty of system failure possibilities is using simulation technique, such as the crude Monte Carlo (CMC). Due to non-frequentative characteristics, a lack of available information, or subjective influences, the usefulness of such methods become evident in many cases within engineering practice [19]. For the computation of extreme event statistics with respect to pollutant loads and environmental effects, the uncertainty in model parameters of deterministic models and the inherent stochastic variability in input variables have to be taken into account [20]. There are two reasons why CMC is difficult to simulate the uncertainty of complex systems: first, condition information of component events generated traditionally by previous experience, historical data and common sense, which does not compensate for interaction between varying failure modes [8, 21]; Second, for the mass noisy problem of complex systems, the complexity of the problem increases exponentially with the number of links and CMC requires a very large simulation effort to estimate the reliability accurately [22]. In this regard it is worthwhile to point out that during the condition monitoring process of a complex system, accurate incorporation of the varying components and subsystems operational condition may necessarily incur a notable increase in the system overall reliability and robustness level.

The information entropy principle acts as a versatile tool on analyzing characters of system failure modes with the least bias beyond conditional constraints. The uncertainty of a single stochastic event A with a known probability $P(A) \neq 0$ plays a fundamental role in information theory. Most conditional moment-constrained information can be expressed in terms of Shannon's original expressions for the entropy [2, 23]. In addition to the single stochastic complete failure events, more important are the incomplete failure mode analysis and condition monitoring of complex systems. The maximum information entropy can be used to assess the uncertainty of failure events of incomplete information. The Kullback-Leibler (K-L) entropy [24] was derived in statistics as an average information measure in a random variable Y for the change of uncertainty in Y from its distribution $p=1$ to distribution $p=0$. It does not require the evaluation of the joint probability density function (pdf) or the conditional pdfs as needed for the mutual entropy [25-26]. The cross-entropy (CE) method [27], based on the concept of the K-L Entropy, is a unified approach to combinatorial optimization, Monte-Carlo simulation and machine learning. Events are considered as abstract concepts and the relations among events are characterized axiomatically [28-32]. In many engineering applications, the system components link and reliabilities are close to 1. The appropriate quantity to measure the performance of a system is then the statistical moment [33]. In such cases the CE method can be used to assess the rare event probability, and we consider more system component operation of relative rankings rather than the exact values of reliabilities.

The rest of the paper is organized as follows. In section 2, a rare event of a redundancy system is formulated for a set of moment constraints in a domain of arbitrary dimension. In section 3, the basic idea of CE as a combinational optimization method is introduced. Section 4 focuses on how the CE method can be used to solve the rare event estimation problem and the combinational optimization problem. We illustrate the effectiveness of the CE method by a number of numerical experiments in section 5. Finally, in section 6, we present our conclusions.

2. Description of a rare-event problem of redundancy system

A quantitative analysis of complex system failure in terms of probability space, i.e., by the distribution of probabilities of events, is difficult. It is recognized [15] that obtaining the component lifetime distribution is the bottleneck. Quantitative and qualitative analysis are both necessary, for the systems and the subsystems of events can be presented by the notion of events and by the appropriate probabilities associated with each of the events [2].

Ambiguities in the condition monitoring of redundancy systems can arise when subsystems' and components' failure events combine and diffuse reciprocally of a redundancy system which has conditional moment-constrained complete or incomplete dynamic failure distributions.

Consider a k out of n vote system which consists of n dynamic subsystems $\mathrm{A}_i\mathrm{B}_j\cdots\mathrm{M}_k$, ($i, j, k = 1, 2, \cdots, n$). Assume any connection in series of out-of-order $\mathrm{A}_i\mathrm{B}_j\cdots\mathrm{M}_k$ has the same system function (Fig. 1).

The relations among component characters are as that illustrated in Fig. 2. Then, the occurrences of random component events are interactive, and the subsystem system failure modes can be regarded as those from one component transfers to another component event according to the alphabet order from A_i to B_j and the last to M_K, where $i, j, k \in (1, n)$.

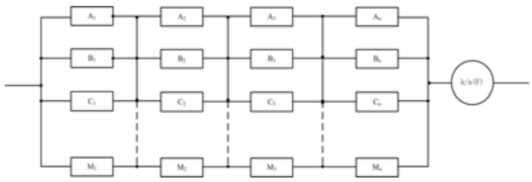

Fig. 1. A series-parallel M × N dimensional redundancy system.

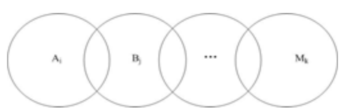

Fig. 2. The interrelations among component characteristics.

2.1 The algebraic structure of a redundancy system

For the above redundancy system of m-dimensional structure with component order up to n domain, we can define the algebraic structure of the events space:

$$E_{ij} \neq \Phi \quad (i=1, 2, \cdots, m;\ j=1, 2, \cdots, n) \tag{1}$$

$$E_{ik}E_{i+1,j} \neq \Phi \quad (i=1, 2, \cdots, k-1) \tag{2}$$

$$E_{ik}E_{lj} = \Phi \quad (l \neq i+1) \tag{3}$$

$$\sum_{i=1}^{m} E_{ij} = I \quad (j=1, 2, \cdots, n) \tag{4}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} P(E_{ij}) \leq 1 \tag{5}$$

- The "Φ" in Eq. (1) means an impossible event;
- Eq. (2) means events E_{ik} and $E_{i+1,k}$ are not necessarily exclusive;
- The fact that E_{ik} and E_{lj} ($l \neq i+1$) are independent is expressed in Eq. (3);
- In Eq. (4), the I denotes a definite dynamic subsystem failure event if any combination of the events E_{ij} in which i is from 1 to m occurs;
- Eq. (5) denotes that the redundancy system is an incomplete system, for only some of the possible events can be found and taken into account.

2.2 Uncertainty associated with events time series of system components

First, a system operational mode analysis is performed to identify all the modes and probabilities of system and subsystems failure events. We give a 2×2 and 3×3 dimensional redundancy system as the foundations for further studying of failure event estimation and reliability optimization of redundancy systems.

2.2.1 The foundation on 2×2 dimensional redundancy system

In Fig. 3, the dynamic subsystems failure modes E^{sub} have $N = (C_2^1 \times C_2^1) = 4$ outcomes. There are

$P(E_1^{sub}) = P(\overline{A_1B_1})$,
$P(E_2^{sub}) = P(\overline{A_1B_2})$,
$P(E_3^{sub}) = P(\overline{A_2B_1})$

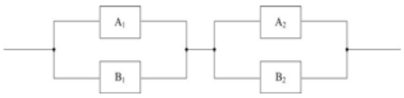

Fig. 3. A two dimension redundancy system.

and

$$P(E_4^{sub}) = P(\overline{A_2 B_2}) .$$

By monitoring the component states, the possible failure time series of the redundancy system have $N^t = (C_2^1 \times C_2^1 + C_4^1 \times C_2^1 \times C_2^1) = 20$ outcomes. The candidate associated failure time series are as follows.

$\overline{A_1 B_1}$; $\overline{A_1 A_2 B_1}$; $\overline{A_1 A_2 B_2}$; $\overline{A_1 B_2 A_2}$; $\overline{A_1 B_2 B_1}$; $\overline{B_1 A_1}$; $\overline{B_1 A_2 A_1}$; $\overline{B_1 A_2 B_2}$;
$\overline{B_1 B_2 A_1}$; $\overline{B_1 B_2 A_2}$; $\overline{A_2 B_2}$; $\overline{A_2 A_1 B_1}$; $\overline{A_2 A_1 B_2}$; $\overline{A_2 B_1 A_1}$; $\overline{A_2 B_1 B_2}$; $\overline{B_2 A_2}$;
$\overline{B_2 A_1 A_2}$; $\overline{B_2 A_1 B_1}$; $\overline{B_2 B_1 A_1}$; $\overline{B_2 B_1 A_2}$;

For the associated failure of components A1, A2 and B2, we consider the possible trajectory of system failure event E^f , that is,

$$\overline{A_1} \to \overline{A_1 B_2} \to \overline{A_1 B_2 A_2} \quad \text{or} \quad \overline{A_2} \to \overline{A_2 B_2} \to \overline{A_2 B_2 A_1}$$

Then the system failure associated probability $P(E^f)$ of components A1, A2 and B2 is

$$\begin{aligned} P(E^f) &= P(\overline{A_1 A_2 B_2}) + P(\overline{A_2 B_2 A_1}) \\ &= P(\overline{A_1}) P(\overline{A_2}) P(\overline{B_2} \mid \overline{A_1 A_2}) \\ &\quad + P(\overline{A_2}) P(\overline{B_2} \mid \overline{A_2}) P(\overline{A_1}) \end{aligned}$$

2.2.2 The foundation on 3×3 dimensional redundancy system

Now consider a three-dimensional redundancy system as another example (Fig. 4). The number of possible components' failure time series (e.g. $\overline{A_1 B_2 C_2 B_3 C_3 C_1 A_3 A_2 B_1}$ in Fig. 5 has $N^t = 9! = 362880$ outcomes, and the dynamic subsystem failure modes have $N^S = (C_3^1 \times C_3^1 \times C_3^1) = 27$ outcomes. For a determinate time series such as $A_3 B_2 C_1 B_3 C_2 B_1 C_3 A_2 A_1$, we can have the following possible dynamic subsystem failure modes:

$$\begin{aligned} P(E_1^f) &= P(\overline{A_3 B_3 C_1}) + P(\overline{B_2 C_3}) + P(\overline{B_1 C_2}) \\ &\quad + P(\overline{A_1 B_1 C_2}) + P(\overline{A_2 B_2 C_3}) \end{aligned} ;$$

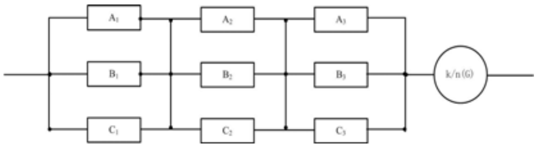

Fig. 4. A three-dimensional redundancy system.

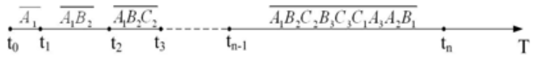

Fig. 5. Time series of system complete failure events.

$$\begin{aligned} P(E_2^f) &= P(\overline{A_3 B_2 C_2}) + P(\overline{B_3 C_1}) + P(\overline{A_2 B_3 C_1}) \\ &\quad + P(\overline{B_1 C_3}) + P(\overline{A_1 B_1 C_3}) \end{aligned} ;$$

$$\cdots\cdots$$

$$\begin{aligned} P(E_n^f) &= P(\overline{A_3 B_2 C_1}) + P(\overline{B_1 C_2}) + P(\overline{A_1 B_1 C_2}) \\ &\quad + P(\overline{A_2 B_3 C_3}) \end{aligned} ;$$

But what we are concerned about the stochastic event time series, such as $A_3 B_2 C_1 B_3 C_2 B_1 C_3 A_2 A_1$, is how to estimate the probability of the dynamic rare-event modes $P(E_1^f)$, $P(E_2^f)$, …, $P(E_n^f)$. The uncertainty analysis of component failure events and their relations can be applied in the pattern recognition of system performance. The following cases in redundancy systems can be achieved by experiences of intuition:

- Different kinds of failure modes may be grouped into the same system failure time series;
- A system failure time series also can have different kinds of system failure modes;
- The component conditions and failure reasons of a redundancy system may be undiscovered;
- The failure of a complex redundancy system may be no more than a rare-event problem.

3. The CE method for Rare-Event simulation

3.1 Kullback-Leibler cross-entropy

Let g and h be two densities with respect to the measure μ on X. The cross-entropy [27] is defined as

$$\begin{aligned} D(\mathrm{g}, h) &= \mathbb{E}_g \ln \frac{g(\mathrm{X})}{h(\mathrm{X})} \\ &= \int g(\mathrm{x}) \ln g(\mathrm{x}) \mu(dx) - \int g(\mathrm{x}) \ln h(\mathrm{x}) \mu(dx) \end{aligned} \quad (6)$$

$D(\mathrm{g}, h)$ is also called the Kullback-Leibler (K-L) divergence, cross-entropy (CE) or relative-entropy.

Meanwhile, $D(g,h)$ is always nonnegative, and it is zero if and only if g and h are exactly the same. This follows from Jensen's inequality (if ϕ is a convex function, such as $-\ln$, then $\mathbb{E}\phi(X) \ge \phi(\mathbb{E}X)$. Namely,

$$D(g,h) = \mathbb{E}_g\left[-\ln\frac{h(X)}{g(X)}\right] \ge -\ln\left[E_g\frac{h(X)}{g(X)}\right] = \ln 1 = 0 \tag{7}$$

The mutual information $M(X,Y)$ of vector X and Y defined as Eq. (8) is related to the CE in the following way:

$$M(X,Y) = D(f, f_X f_Y) = \mathbb{E}_f\left[\ln\frac{f(X,Y)}{f_X(X)f_Y(Y)}\right] \tag{8}$$

where f is the joint pdf of (X, Y), f_X and f_Y are the marginal pdfs of X and Y, respectively. The mutual information can be viewed as the CE measure, i.e., the "distance" between the joint pdf f of X and Y and the product of their marginal pdfs f_X and f_Y under assumption that the vectors X and Y are independent.

3.2 The Cross-Entropy method

The cross-entropy method [27], which is based on an associated CE minimization, is a well known technique for estimating probabilities of rare events. In recent years the CE method has been successfully applied to a wide range of discrete optimization tasks [28, 29, 31]. In the field of rare-event simulation, the CE method is used in conjunction with importance sampling (IS) and it provides a simple and fast adaptive procedure for estimating the optimal reference parameters in the IS. The two steps of the CE algorithm are:

(1) Importance sampling generating. Describe the distributing function of independent variables of the objective function, which usually generates a random permutation. Establish convergence of the algorithm under much weaker conditions.

(2) Adaptive parameter updating. Update the parameters of this permutation to obtain better system reliabilities in the next iteration. A stopping rule should be set in advance when the desired value of results is researched at some iteration t. The aim of this step is to prove convergence for a finite sample with the emphasis on the complexity and the speed of convergence under the suggested stopping rules.

In the field of combination optimization, the CE method can be readily applied by first translating the underlying optimization problem into an associated estimation problem (ASP) which typically involves rare event estimation. Suppose we wish to maximize some "performance" function $S(x)$ over all elements (states) x in some set X. Let us denote the maximum by γ^*, thus

$$\gamma^* = \max_{x \in X} S(x) \tag{9}$$

To proceed with CE, we first randomize our deterministic problem by defining a family of pdfs $f(\cdot;u)$ on the set X. Next, we associate with Eq. (10) the estimation of

$$l(\gamma) = P_u(S(X) \ge \gamma) = \mathbb{E}_u I_{\{S(X)\ge\gamma\}} \tag{10}$$

A viable method to estimate l in Eq. (10) is to use crude Monte Carlo (CMC) simulation, draw a random sample $X_1,\cdots,X_N$ from the distribution of X via the Eq. (11) as the unbiased estimation of l.

$$\hat{l} = \frac{1}{N}\sum_{i=1}^{N} I_{\{S(X)\ge\gamma\}} \tag{11}$$

However, for large γ the probability of l is very small; CMC requires a very large N to obtain a small relative error. The CE method can be used in such situations efficiently. The main idea [32] of the CE method for rare event simulation and optimization can be stated as follows. Eq. (9) is called an associated stochastic problem (ASP). Here, X is a random vector with pdf $f(\cdot;u)$, for some $u \in V$ and γ is a known or unknown parameter. Note that there are in fact two possible estimation problems associated with Eq. (10). For a given γ we can estimate l, or alternatively for a given l we can estimate γ which is the root of Eq. (10). The CE method solves the problem efficiently by making adaptive changes to the probability density function according to the K-L cross-entropy, thus creating a sequence $f(\cdot;u)$ and $f(\cdot;v)$ of pdfs that are "steered" in the direction of the theoretically optimal density $f(\cdot;v^*)$ corresponding to the degenerate density at an optimal point. In fact, the CE method generates an adaptive updating of γ_t and v_t sequence -{ γ_t, v_t }, which converges quickly to a small neighborhood of the optimal { γ^* , v^* }. For the main applications of the CE method, updating of parameter vector γ and v can be formulated as the following two algorithms.

3.3 A Rare-Event simulation example

The CE has its origins in an adaptive algorithm for rare event simulation, which transforms the original deterministic events into an associated stochastic process analysis [34, 35]. For the event time series described in section 2.2, we estimate the short path to simulate the rare event as an example and illustrate the basic steps of the CE method.

Employ an auxiliary weighted graph of Fig. 6, with random weights $X_1,\cdots,X_n$. Suppose the weights are independent and exponentially distributed random variables with means $u_1,\cdots,u_n$ respectively.
Denote the pdf of X by $f(\cdot;u)$. Let $S(X)$ be the total length of the shortest path from node A to node B, then

$$f(x;u) = \exp\left(-\sum_{j=1}^{n}\frac{x_j}{u_j}\right)\prod_{j=1}^{n}\frac{1}{u_j} \tag{12}$$

To estimate S(X) from simulation l^*, we have

$$l = P(S(X)\ge\gamma) = \mathbb{E}I_{\{S(X)\ge\gamma\}} \tag{13}$$

For some large fixed ɣ and a small relative error (RE), a better way to perform the simulation is to use importance sampling as

$$l = \int I_{\{S(X)\ge\gamma\}}\frac{f(x)}{g(x)}g(x)dx = \mathbb{E}_g I_{\{S(X)\ge\gamma\}}\frac{f(x)}{g(x)} \tag{14}$$

The likelihood ratio W(x) is

$$W(X_i) = \frac{f(x_i)}{g(x_i)} \tag{15}$$

To restrict g such that $X_1,\cdots,X_n$ are independent and exponentially distributed with means $v_1,\cdots,v_n$, we have

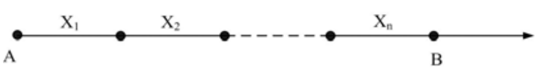

Fig. 6. Shortest path from A to B.

$$\begin{aligned} W(x;u,v) &= \frac{f(x;u)}{f(x,v)} \\ &= \exp\left(-\sum_{j=1}^{n}x_j\left(\frac{1}{u_j}-\frac{1}{v_j}\right)\right)\prod_{j=1}^{n}\frac{v_j}{u_j} \end{aligned} \tag{16}$$

Using the above stochastic counterpart method by minimizing the Kullback-Leibler distance of $f(\cdot;u)$ and $f(\cdot;v)$, we have the following updating formula [34]

$$v_{t,j} = \frac{\sum_{i=1}^{N} I_{\{S(X)\ge\gamma_t\}}W(X_i;u,v_{t-1})X_{ij}}{\sum_{i=1}^{N} I_{\{S(X)\ge\gamma_t\}}W(X_i;u,v_{t-1})} \tag{17}$$

For some fixed γ and a small relative error, the problem is how to select a v to give estimate of l under some fixed relative error (RE) which is

$$RE = \frac{\sqrt{Var(l)}}{l^*} \tag{18}$$

Therefore, to solve the above event simulation problem, first we need to generate some viable stochastic samples. Then the CE algorithm can update the parameters at each of iterations. For convenience, we define an auxiliary function $T(x,\xi)$ where ξ is presented in another way as $\xi = (\xi_1,\cdots,\xi_{m_i})$ and m_i is the number of random variables. Then, we calculate the following function which is the minimum length of $S(X)$:

$$U_1 : x \to \mathbb{E}\left[T(x,\xi)\right] \tag{19}$$

The CE algorithm for obtaining $v_{t,j}$ and the probability of l can be developed as follows.

Algorithm 1: CE algorithm for Rare-Event simulation

Step 1. Define $\hat{v}_0 = u$, $T = 0$. Set $t = 1$ (iteration counter).

Step 2. Generate random samples $X_1,\cdots,X_N$ from step 2.1 to step 2.6.

Step 2.1. Set $u(x) = 0$, $i = 1$.

Step 2.2. Generate $\xi = (\xi_1,\cdots,\xi_n)$ from the distribution function $f(\cdot;\hat{v}_{t-1})$.

Step 2.3. Calculate U_1 according to Eq. (19), that is $u(x) \leftarrow u(x) + T(x,\xi)$.

Step 2.4. $S_{(i)} = u(x)$.

Step 2.5. Set $u(x) = 0$, $i = i + 1$.

Step 2.6. Repeat from step 2.2 to step 2.5 N times.

Step 3. Calculate the performance $S(\mathrm{X}_i)$ for all i, and order them from smallest to biggest, $S_{(1)} \le ... \le S_{(N)}$. Let $\hat{\gamma}_t$ be the sample $(1-\rho)$ -quantile of performances: $\gamma_t = S_{\lceil (1-\rho)N \rceil}$, provided this is less than γ. Otherwise, put $\hat{\gamma}_t = \gamma$.

Step 4. For $j = 1,\cdots,n$, use the same sample to calculate

$$v_{t,j} = \frac{\sum_{i=1}^{N} I_{\{S(\mathrm{X}) \ge \gamma_t\}} \mathrm{W}(\mathrm{X}_i;\mathrm{u},\mathrm{v}_{t\text{-}1})\mathrm{X}_{ij}}{\sum_{i=1}^{N} I_{\{S(\mathrm{X}) \ge \gamma_t\}} \mathrm{W}(\mathrm{X}_i;\mathrm{u},\mathrm{v}_{t\text{-}1})}$$

Step 5. If $\hat{\gamma}_t = \gamma$ then proceed to step 5; otherwise set $t = t+1$ and reiterate from step 2.

Step 6. Let T be the final iteration. Generate a sample $\mathrm{X}_1,\cdots,\mathrm{X}_{N_1}$ according to the pdf $f(\cdot;\hat{\mathrm{v}}_T)$, and estimate l via the important sampling estimator

$$\hat{l} = \frac{1}{N_1}\sum_{i=1}^{N_1} I_{\{S(\mathrm{X}) \ge \gamma_t\}} \mathrm{W}(\mathrm{X}_i;\mathrm{u},\hat{\mathrm{v}}_T)$$

4. The uncertainty estimation of redundancy system

Starting from the rare-event problem defined in Section 2.2 and if $\mathrm{C_i}$ can transfer to $\mathrm{A_j}$ ($i, j = 1,2,3$), the problem can be regarded as a multi-extremal traveling salesman problem. Fig. 7 illustrates such a problem. Consider a weighted graph G with $m \times n$ nodes, labeled $(\mathrm{A}_1,\mathrm{A}_2,\cdots,\mathrm{A}_\mathrm{n})$; $(\mathrm{B}_1,\mathrm{B}_2,\cdots,\mathrm{B}_\mathrm{n})$; $\cdots$; $(\mathrm{M}_1,\mathrm{M}_2,\cdots,\mathrm{M}_\mathrm{n})$. The nodes represent cities, and the edges represent the roads connecting the cities. Each edge from *i* to *j* has weight or cost $\mathrm{C_{ij}}$, representing the length of the road. Given the starting city and the terminating city, one has to estimate a possible shortest tour that visits all the cities exactly once (such as from $\mathrm{A_3}$ to $\mathrm{C_3}$).

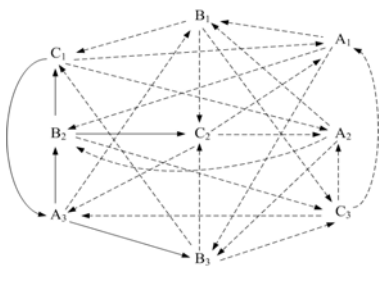

Fig. 7. Shortest path from A3 to C3.

The possible shortest path is a moment-constrained combination optimization problem, while the waiting time should be calculated. Consider a binary vector $\mathrm{y} = (\mathrm{y}_1,\mathrm{y}_2,\cdots,\mathrm{y}_{27})$. Suppose that we do not know which components of y are 0 and which are 1. Input $\mathrm{x} = (\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_{27})$ to reconstruct y by maximizing the function S(x) and minimizing the Euclidean distance of Var(Pt).

$$\max \quad S(\mathrm{x}) = 27 - \sum_{j=1}^{27} \left| x_j - y_j \right| \tag{20}$$

s.t.

$$\min \quad \mathrm{Var}(\mathrm{P_t}) = \left\| \mathrm{P_t} - \mathrm{P}^* \right\| = \sqrt{(\mathrm{P}_{\mathrm{t},i} - \mathrm{P}^*)^2} \tag{21}$$

Here we design stochastic samples generated as follows. Let y represent a random path which is

$$y = \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{m_1 1} & \cdots & y_{m_i n} \end{pmatrix}$$

where y_{ij} are independent Bernoulli random variables. We calculate the function which is the minimum of the performance $S(\mathrm{X})$

$$U_2 : y \rightarrow E\left[S(y \mid \mathrm{X})\right] \tag{22}$$

Then a stochastic simulation sample can be developed. We then evaluate the performances of these probability vectors for the CE algorithm 2, constrained condition of Eq. (21), with the best performance as our final solution to the problem. The CE algorithm for the uncertainty estimation of redundancy systems is outlined below, which is similar to the rare-event simulation algorithm described in section 3.3.

Algorithm 2: CE algorithm for uncertainty estimation of redundancy systems

Step 1. Set $t = 1$ (iteration counter). Start with some $\hat{p}_0$ such as

$$\hat{p}_0 = \begin{pmatrix} 0.5 & \ldots & 0.5 \\ \vdots & \ddots & \vdots \\ 0.5 & \cdots & 0.5 \end{pmatrix}$$

Step 2. Draw a sample $\mathrm{X}_1,\cdots,\mathrm{X}_N$ from step 2.1 to step 2.6.

Step 2.1. Set $U_2(y)=0$, $i=1$.

Step 2.2. Generate $y=\begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{m_1 1} & \cdots & y_{m_i n} \end{pmatrix}$ from the Bernoulli vectors with success probability vector $\hat{p}_{t-1}$.

Step 2.3. $U_2(y) \leftarrow U_2(y)+S(y\,|\,\mathrm{X})$.

Step 2.4. $S_{(i)}=U_2(y)$.

Step 2.5. Set $U_2(y)=0$, $i=i+1$.

Step 2.6. Repeat from step 2.2 to Step 2.5 N times.

Step 3. Calculate the performance $S(\mathrm{X}_i)$ for all i, and order them from smallest to biggest, $S_{(1)} \le \cdots \le S_{(N)}$. Let $\hat{\gamma}_t$ be the sample $(1-\rho)$ - quantile of performances: $\gamma_t = S_{\lceil (1-\rho)N \rceil}$, provided this is less than γ. Otherwise, put $\hat{\gamma}_t=\gamma$.

Step 4. Use CE method to generate an adaptive updating of $\hat{\mathrm{P}}_t$, via the following formula [27]

$$\hat{p}_{t,j} = \frac{\sum_{i=1}^{N} I_{\{S(\mathrm{X}_i)\ge \gamma_t\}} I_{\{X_{ij}=1\}}}{\sum_{i=1}^{N} I_{\{S(\mathrm{X}_i)\ge \gamma_t\}}}$$

where $j=1,\cdots,n$, $\hat{\mathrm{P}}_t=(\hat{p}_{t,1},\cdots,\hat{p}_{t,n})$ and $\mathrm{X}_i=(X_{i1},\cdots,X_{in})$

Step 5. Reiterate from step 2 until the stopping criterion of min Var(Pt) is met.

In the situation described in Section 2.2.2, only one optimal time series exists but where many time series can be the possible dynamic subsystem failure modes, whose reliabilities of component links are very close to 1. The probability vector of the CE algorithm 2 for the uncertainty estimation could oscillate, and this would increase the computational effort. In such cases, consider an auxiliary parameter β and a certain threshold δ in Fig. 7, say $\beta=0.05$, $\delta=3$, and then in Fig. 7 the solid lines probabilities are 1 and the candidates broken lines' probabilities lie in the ranges $[1-\beta,1]$. Algorithm 2 may terminate once the number of probabilities that lie between $[1-\beta,1]$ falls below the threshold δ. We then can generate all the candidate probability vectors according to probability vectors of the time series.

5. Numerical results

The two algorithms developed above are hybrid algorithms in which the Monte Carlo simulations and CE optimization algorithms are combined. A general computational framework of event estimation and condition monitoring of redundancy systems can be developed from the two algorithms. In this section we give an example of the framework for the event estimation problem. It mainly includes how to generate stochastic shortest path by the shifted exponential distribution. For simplification, the following illustrates the procedure of a shortest path estimation problem.

The shortest path in a stochastic network can be defined as

$$S(\mathrm{X}) = \min_{j=1,\cdots,\mathfrak{M}} \sum_{i\in\mathfrak{M}_j} X_i \tag{23}$$

- $\mathfrak{M}$ is the number of complete paths from a source to a sink;
- $\mathfrak{M}_j$ is the j-th number of complete paths;
- $\mathrm{X}=(X_1,\cdots,X_n)$ is the reference vectors of components;
- X_i, $i=1,\cdots,n$, represent the weights of the links;
- $S(\mathrm{X})$ is the total length of the shortest path from a source node to a sink node.

Assume $X \sim \mathrm{Exp}(u^{-1})$ and the shortest path exceeds some fixed γ, then we have

$$l = \mathbb{E}_u I_{\{S(\mathrm{X})\ge\gamma\}} = \int_{\gamma}^{\infty} u^{-1} e^{-xu^{-1}} dx = e^{-\gamma} u^{-1} \tag{24}$$

and the optimal importance sampling density of X becomes

$$\begin{aligned} g^*(x) &= I_{\{x\ge\gamma\}} u^{-1} e^{-xu^{-1}} e^{\gamma u^{-1}} \\ &= I_{\{x\ge\gamma\}} u^{-1} e^{-x(x-\gamma)u^{-1}} \end{aligned} \tag{25}$$

Eq. (25) suggests that $g^*(x)$ is the shifted exponential distribution of X. Thus, we can use a virtual path to implement the shifted processing from pdf $f(\mathrm{x};\mathrm{u})$ to the importance sampling density $f(\mathrm{x};\mathrm{v})$. That is from

$$f(\mathrm{x};\mathrm{u}) = \exp\left(-\sum_{j=1}^{n} \frac{x_j}{u_j}\right) \prod_{j=1}^{n} \frac{1}{u_j} \tag{26}$$

to

$$f(\mathrm{x};\mathrm{v}) = \exp\left(-\sum_{j=1}^{n} \frac{x_j}{v_j}\right) \prod_{j=1}^{n} \frac{1}{v_j} \tag{27}$$

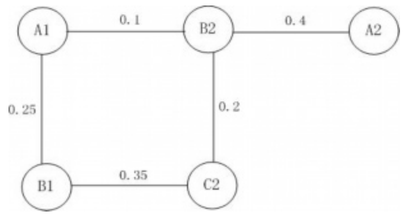

Fig. 8. Shortest path simulation from A1 to A2.

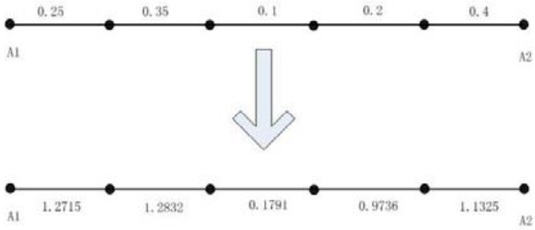

Fig. 9. The probability density updating in the virtual path from A1 to A2.

The CE method has the high efficiency for simulating a rare-event problem and it can be used in simulating various combinational optimization problems. In practice, most combinational optimization problems can be transferred as a rare-event model and can be solved by the above shortest path computational framework. For example, to estimate the distribution of paths from a node to another, say *i* to *j*, in a traveling salesman problem model with a performance less than or equal to some γ, we can count the number of paths going from i to j which has a performance less than or equal to γ, and divide the number by the total number of paths passing through node i whose performance is less than or equal to γ. Then, it can be simulated as a rare-event problem by the CE algorithm. The principal outcome of the CE approach is the construction of a random sequence of solutions which converges probabilistically to the optimal or near-optimal solution of combinational optimization problems.

As a numerical example of Algorithm 1, we give the statistical moments of a system whose links are relative ranking failure events. Fig. 8 assumes that the system failure mode is $\overline{A1A2}$, while its failure time series include $\overline{A1B2A2}$ and $\overline{A1B1C2B2A2}$. Suppose we need to estimate the probability that the minimum path from A1 to A2 is greater than γ =2.

Given the initial parameter u= (0.25, 0.35, 0.1, 0.2, 0.4), with n=5, N= 1000, $\rho = 0.1$, and that the final iteration of constrained condition in Eq. (17) has relative error RE=0.03. Fig. 9 illustrates the probability density shifted processing from u= (0.25, 0.35, 0.1, 0.2, 0.4) to the reference parameter v= (1.2715, 1.2832, 0.1791, 0.9736, 1.1325) corresponding to the virtual path with the source node A1 to the sink node A2.

Table 1. Evolution of the sequence $\{(\gamma_t, v_t)\}$.

t	γ_t	v_t				
0		0.2500	0.3500	0.1000	0.2000	0.4000
1	0.5527	0.4843	0.5487	0.0976	0.3662	0.5063
2	0.9609	0.7345	0.8307	0.1318	0.5000	0.7037
3	1.4043	0.9837	1.1704	0.1518	0.7004	1.0775
4	1.8354	1.3209	1.4069	0.0773	0.8168	1.0486
5	2.0000	1.2715	1.2832	0.1791	0.9736	1.1325

Table 1 gives the detailed evolution of the CE algorithm with the adaptive updating sequence $\{(\gamma_t, v_t)\}$ of the reference parameter v_t and the level parameter γ_t.

Using the estimated optimal parameter vector v_5 = (1.2715, 1.2832, 0.1791, 0.9736, 1.1325), the final step with N1=105, we get an estimate of $\hat{l} \approx 1.18 \cdot 10^{-5}$ with an estimated RE of 0.03. This result was computed in less than half a second on an AMD processor with 512 M RAM.

6. Conclusions

The idea underlined in the paper is to relate the uncertainties of failure modes of subsystem events and the uncertainty of the entire system. By adequate partitioning of the event space, the exclusive or inclusive relation analysis of component events, which are conditional moment-constrained information, can provide a better insight into the system reliability. This paper discusses the uncertainty estimation problem of reliability redundancy because redundancies are inherent in the majority of complex engineering systems. The CE method, which is a well known stochastic simulation approach, is introduced to solve the reliability optimization problems of redundancy systems. For eliminating the uncertainties of system failure possibilities, the CE method algorithm employs a stochastic counterpart method, which transforms the original deterministic events into an associated stochastic one, and handles the relative rankings of events for solving the complex constrained combinational optimization problem.

A general computational framework of redundancy system failure diagnosis and condition monitoring can be developed based on the CE method. The traditional probabilistic engineering system analysis is difficult to solve the uncertainty problem of the random and deterministic events. However, the CE method, which is a convex method based on probabilistic and non-probabilistic set-theory, considers events as abstract concepts and the relations among events are characterized axiomatically. By using the shifted probability distribution function, it is obvious that the CE method can deal successfully with both the rare-event simulation and the uncertainty estimation of redundancy systems. For highly redundant and robust systems, in which most components link reliabilities are close to 1, relative rankings analysis has become more important than the exact values of variable reliabilities. The concept of the virtual path corresponding to relative ranking probabilities and the shifted probability density function illustrates a simple framework and an efficient strategy for developing the mechanism of generating random data samples. It will dramatically expand the applications of the CE method on reliability optimization of redundancy systems.

Acknowledgment

This research was partially supported by the National Natural Science Foundation of China under the contract number 50775026, the Specialized Research Fund for the Doctoral Program of Higher Education of China under the contract number 20060614016. The authors would like to thank the referees and the editors for their comments and constructive suggestions. We are also grateful to Thomas Taimre for discussing with us on implementing the CE Matlab codes.

Nomenclature

ϕ : Convex function in general
Φ : Impossible event
ξ : Random sample parameter
$\underline{\mu}$: Base measure
$\overline{A}$: Failure event of component A
C_n^k : Number of combination (*n; k*)
D : Kullback-Leibler cross-entropy
E : Random event in general
E_{ij} : Failure event of system component
E^{sub} : Subsystems failure events
E^f : System failure event
$\mathbb{E}$: Expectation
$\mathbb{E}_u$: Expectation taken with respect to u
f : Probability density function
ln : Natural logarithm
I : Definite failure event of system
I_A : Indicator function of event A
l : Length parameter of event
$l(\gamma)$: Performance evaluation of event
M : Mutual information
m_i : Number of random variables
$\mathfrak{M}$: Number of complete paths
$\mathfrak{M}_j$: j-th complete path
N : Number of samples for a normal step
N_1 : Number of samples for the final step
N^S : Number of system failure modes
N^t : Number of failure time series
$P(E_{ij})$: Probability of E_{ij}
P_u : Probability taken with respect to u
S : Performance function of length
$S_{(i)}$: i-th order statistic
T : Auxiliary function of S
u : Nominal reference parameter (vector)
$U:x$: Sample function
v : Reference parameter (vector)
W : Likelihood ratio
x, y : Vectors
X,Y : Random vectors/matrices
$\hat{l}$, $\hat{p}$: Estimated reference parameters
ρ : Rarity parameter
γ : Level parameter of event
$\gamma*$: CE optimal level parameter of event
β : Auxiliary parameter
δ : Threshold
Exp : Exponential distribution
RE : Relative error
Var : Variation error

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Gui-Bao Wang is currently a Ph. D. student studied at the University of Electronic Science and Technology of China. He received a B. S. degree in Manufacturing Engineering from Beijing University of Aeronautics and Astronautics, and an M. E. in Mechanical Engineering form Tsinghua University. He has been employed as a military representative by PLA. His main research interests include complex system optimization, information entropy theory and the Cross-Entropy method.

Hong-Zhong Huang is a full professor and the Dean of the School of Mechanical, Electronic, and Industrial Engineering, University of Electronic Science and Technology of China. He received a Ph. D. in Reliability Engineering from Shanghai Jiaotong University, China. He has published 150 journal papers and 5 books in the field of reliability engineering. He has held visiting appointments at several universities in the USA, Canada, and Asia. He received the Golomski Award from the Institute of Industrial Engineers in 2006. His current research interests include system reliability analysis, warranty, maintenance planning and optimization, computational intelligence in product design.

Yu Liu received the B.E. and M.E. degrees in Mechatronics Engineering from the University of Electronic Science and Technology of China. He is currently working towards a Ph.D in the School of Mechatronics Engineering at University of Electronic Science and Technology of China, and is pursuing his Ph.D research in the Mechanical Engineering Department at Northwestern University, U.S.A, as a pre-doctoral visiting fellow. His research interests include design under uncertainty, reliability of multi-state systems, maintenance decisions, and optimization.

Xiaoling Zhang is currently pursuing the Ph.D. degree at the School of Mechanical, Electronic, and Industrial Engineering, University of Electronic Science and Tech-nology of China. Her main research interests include the analytical target cascading in complex systems, robust design and system reliability optimization.

Zhonglai Wang is a Ph.D. candidate in the School of Mechanical, Electronic, and Industrial Engineering, University of Electronic Science and Technology of China. He has been a visiting scholar in the Department of Mechanical and Aerospace Engineering, Missouri University of Science and Technology from 2007. to 2008.. His research interests include reliability-based design and robust design.